

IX. *On Infinite Series.* By Edward Waring, M. D. F. R., S. Luca-
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1. **M**ERCATOR first published the continuation of the common method of division to an infinite series of terms proceeding according to the dimensions of a variable quantity; NEWTON did the same for the common method of extraction of roots. Dr. BARROW before applied the same principles in some easy examples to find the asymptotes of curves.

2. The methods of division and extraction of roots were long before taught; but the continuation of them *in infinitum* would have been useless, as the areas of curves, whose ordinates are $ax^{\frac{n}{m}}$ (where x denotes the absciss, and a , n , and m invariable quantities) had not been discovered many years before the time of MERCATOR's Publication, and consequently it would have been of little use to transform an ordinate or fluxion, whose area or fluent is unknown, into another form, of which the area, &c. is equally unknown.

3. Sir ISAAC NEWTON extended the rule for raising a binomial (to any affirmative power) to negative powers, the extraction of roots and fractional indexes, by applying the law of the series for affirmative powers to them, and continuing it *in infinitum*. M. DE MOIVRE extracted the root, &c. of a multinomial by a series of a similar nature; but these methods will

will only apply in the most simple cases, when not more than one root is to be extracted. In every complicate case (*viz.* the extraction of roots of quantities which involve the roots of compound quantities) of irrational quantities, recourse must be had to the old methods of multiplication, division, and extraction of roots.

4. If a root of a complicate irrational quantity be required by a series proceeding according to the dimensions of x ; first reduce all the irrational quantities contained under the root by multiplication, division, and extraction of roots into series proceeding according to the dimensions of x , so that the terms of the least dimensions be constituted first, if an ascending series be required, and so on; and the contrary, if a descending; then add the several sums together, and extract the root of the resulting sum by a series which proceeds according to the dimensions of x , and it will be the root required.

Ex. Let the value of the quantity $\sqrt[m]{(x^{-2} + a + \sqrt{b + cx} \times \sqrt{(d + ex + \sqrt{(f + gx^2)) + \sqrt{(b + kx^2))})}$ be required by an ascending series; first, extract the innermost root $\sqrt{(f + gx^2)} = f^{\frac{1}{2}} + \frac{gx^2}{2f^{\frac{1}{2}}} - \&c.$ which add to the quantity $d + ex$ contained under the same root (m), and place the terms of the sum according to the dimensions of x , and it becomes $(d + f^{\frac{1}{2}}) + ex + \frac{gx^2}{2f^{\frac{1}{2}}} - \&c.$ of which extract the (m) root, and there results

$$\left. \begin{aligned} (d + f^{\frac{1}{2}})^{\frac{1}{m}} + \frac{1}{m} \times \frac{1-m}{d + f^{\frac{1}{2}}^{\frac{1}{m}}} \times ex + \frac{1}{m} \times \frac{1-m}{2m} \times \frac{1-2m}{d + f^{\frac{1}{2}}^{\frac{1}{m}}} \times e^2 x^2 + \\ + \frac{1}{m} \times \frac{1-m}{d + f^{\frac{1}{2}}^{\frac{1}{m}}} \times \frac{g}{2f^{\frac{1}{2}}} \end{aligned} \right\}$$

&c. multiply this series into $b + cx$, and thence is derived

$$\left. \begin{aligned} b \times \overline{d+f^{\frac{1}{2}m}}^{\frac{1}{m}} + \frac{1}{m} b \times \overline{d+f^{\frac{1}{2}m}}^{\frac{1-m}{m}} \times e \\ + c \times \overline{d+f^{\frac{1}{2}m}}^{\frac{1}{m}} \end{aligned} \right\} x + \&c. = P; \text{ extract the root}$$

$$\sqrt{b+kx^2} = b^{\frac{1}{2}} + \frac{kx^2}{2b^{\frac{1}{2}}} + \&c. = Q; \text{ add the three quantities P, Q,}$$

and $x^{-2} + a$ contained under the same root $\sqrt[3]{}$ together, and the series resulting, whose terms are constituted according to the dimensions of x , will be $x^{-2} + (a + b \times \overline{d+f^{\frac{1}{2}m}}^{\frac{1}{m}} + b^{\frac{1}{2}} = A) + (\frac{1}{m} b \times \overline{d+f^{\frac{1}{2}m}}^{\frac{1-m}{m}} \times e + c \times \overline{d+f^{\frac{1}{2}m}}^{\frac{1}{m}} = B) x + \&c.$; of which extract the cube root $\sqrt[3]{}$, and it will be $x^{-\frac{2}{3}} + \frac{1}{3}Ax^{\frac{4}{3}} + \frac{1}{3}Bx^{\frac{7}{3}} + \&c.$ the root required.

5. The principal use of reducing quantities into series proceeding according to the dimensions of the variable quantity is as before mentioned for finding the area of a curve from its ordinate; or, which corresponds, the integral from its nascent or evanescent increment; but the serieses deduced should converge, otherwise from them cannot be found the area or integral. In the *Meditationes Analyticae* a method was first published of finding when these series will converge and when not, *e. g.* the series $a + bx + cx^2 + dx^3 + \&c. = \int (A + Bx + Cx^2 + Dx^3 + \&c.)^m \times \dot{x} = P$ will converge when (x) either affirmative or negative is less than the least root (α) of the equation $A + Bx + Cx^2 + Dx^3 + \&c. = 0$, if the roots are possible. A similar rule is given when some of the roots are impossible. The series will diverge when x is greater than α , and the cases are given in which it will converge when $x = \alpha$. The series descending according to the dimensions of x will converge when x is greater than the greatest root of the equation, &c. These principles are further

ther applied in the same Book to complicate irrational algebraical functions of x , &c.

Hence most commonly the series for the area contained between two ordinates, or integral between two different increments deduced by the common method will diverge; on which account, in the same Book, is given a method by interpolation of finding the area or integral contained between any two different values of x by converging series, if the area, &c. is finite.

6. To find whether a given value ($+a$) is less than the least affirmative or negative root (x) of a given algebraical equation $A + Bx + Cx^2 + Dx^3 + \&c. = 0$, if all its roots are possible; transform the equation into another, whose root z is the reciprocal of the root $x = \frac{1}{z}$ of the given equation, and for z in the resulting equation write respectively $v + a$ and $v - a$; and if from the former substitution all the terms become negative or affirmative, and from the latter they become alternately negative and affirmative, then will a be less than the least root of the given equation. If in the same manner, in the given equation for x be substituted $v + a$ and $v - a$, and the terms result as before, then will (a) be greater than the greatest root affirmative or negative of the given equation.

7. When the integral of an algebraical quantity, whose increments are finite, is required; first, by the method given in *Medit. Analyt.* investigate the integral in finite terms, if it can be expressed by them; but if not, reduce it into infinite serieses of which the integral of each of the terms can be found, and also the serieses for finding the integral contained between the two different given values of the variable quantity may converge.

Serieses of this kind have been given in the *Medit. Analyt.* and innumerable of a like kind may be added for finding integrals

integrals by converging series either ascending or descending, of which the given increments are either finite or evanescent.

7. 2. It may be observed, that generally the particular case of which the increments are nascent or evanescent may be deduced from the general, in which the increments are finite; and consequently in many cases the general will, *mutatis mutandis*, correspond to the particular; e. g. 1. the integral cannot be expressed in finite algebraical terms, when any factor in the denominator of the increment has not a successive correspondent one; which is analogous to the case of the simple divisor in the denominator of a fluxion published in the Quadrature of Curves. 2. Nor can it be expressed by the above-mentioned terms, when the dimensions of the variable quantity in the denominator exceed its dimensions in the numerator by unity, which corresponds to a similar case in fluxions first given in *Medit. Analyt.* To these may several others be added.

8. Let the fluent of the fluxion $(A + Bx^n + Cx^{2n} + \&c.)^m \times x^{\theta-1} \dot{x} = ax^{\theta} + bx^{\theta+n} + cx^{\theta+2n} + \&c. = (A + Bx^n + Cx^{2n} + \&c.)^{m+1} \times \frac{ax^{\theta} + \beta x^{\theta+n} + \gamma x^{\theta+2n} + \&c.}{ax^{\theta} + \beta x^{\theta+n} + \gamma x^{\theta+2n} + \&c.} = a'x^{\lambda} + b'x^{\lambda-n} + c'x^{\lambda-2n} + \&c. = (A + Bx^n + Cx^{2n} + \&c.)^{m+1} \times \frac{a'x^{\lambda} + \beta'x^{\lambda-n} + \gamma'x^{\lambda-2n} + \&c.}{ax^{\theta} + \beta x^{\theta+n} + \gamma x^{\theta+2n} + \&c.}$

1. If the infinite ascending series $ax^{\theta} + \beta x^{\theta+n} + \gamma x^{\theta+2n} + \&c.$ converges, then will the series $ax^{\theta} + bx^{\theta+n} + cx^{\theta+2n} + \&c.$ converge, and nearly in the same ratio; and *vice versa*, if the former diverges, the latter will also diverge. In the same manner if the infinite descending series $a'x^{\lambda} + \beta'x^{\lambda-n} + \gamma'x^{\lambda-2n} + \&c.$ converges, then will the series $a'x^{\lambda} + b'x^{\lambda-n} + c'x^{\lambda-2n} + \&c.$ converge; and if the former diverges, then will the latter also diverge; and in all the cases nearly in the same ratio, except only when their convergency is the least.

2. If the series $ax^0 + bx^{\theta+n} + cx^{\theta+2n} + \&c.$ converges, then will the series $a'x^\lambda + b'x^{\lambda-n} + c'x^{\lambda-2n} + \&c.$ diverge, unless in cases of the slowest convergency, where $x = \pm v \sqrt[\theta]{\pm 1}$, and all the roots of the given equation are of the formula $\pm v \sqrt[\theta]{\pm 1}$; when $x = \pm v \sqrt[\theta]{\pm 1}$, the successive terms of the infinite series deduced from algebraical quantities by the preceding method will ultimately, that is, at an infinite distance come more near to a ratio of equality with each other than any assignable difference.

3. If the fluxion be $(A + Bx^n + Cx^{2n} \dots x^n)^m \times x^{\theta-1} \dot{x}$, then will $\lambda = mrn + \theta$, and $\lambda' = \theta - rn$.

Many more propositions concerning infinite series and their convergency are given in the *Medit. Analyt.*

4. Let the fluxion be $\overline{a + bx^n}^m \times x^n \dot{x}$, of which find the fluent in a series ascending according to the dimensions of x^n , and it

$$\begin{aligned}
 &\text{will be } a^m \times x^{b+1} \left(\frac{1}{b+1} + \frac{m}{b+n+1} \times \frac{b}{a} x^n + m \cdot \frac{m-1}{2(b+2n+1)} \times \frac{b^2}{a^2} x^{2n} \right. \\
 &+ m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3(b+3n+1)} \times \frac{b^3}{a^3} x^{3n} + \&c. \left. \right) = \frac{b^m x^{mn+b+1}}{mn+b+1} + \\
 &\frac{mb^{m-1} a x^{mn+b+1-n}}{mn+b+1-n} + m \cdot \frac{m-1}{2} \times \frac{b^{m-2} a^2 x^{mn+b+1-2n}}{mn+b+1-2n} + \&c. = \overline{a + bx^n}^{m+1} \\
 &\times \left(\frac{x^{b+1}}{b+1a} - \frac{m+1(n)+b+1}{b+1 \cdot b+1+n} \times \frac{b}{a^2} x^{b+n+1} + \frac{(m+1)(n)+b+1 \cdot (m+2)(n)+b+1}{b+1 \cdot b+1+n \cdot b+1+2n} \right. \\
 &\times \frac{b^2}{a^3} \times x^{b+2n+1} - \frac{(m+1 \cdot n+b+1) \cdot (m+2n+b+1) \cdot (m+3n+b+1)}{b+1 \cdot b+1+n \cdot b+1+2n \cdot b+1+3n} \times \\
 &\left. \frac{b^3}{a^4} x^{b+3n+1} + \&c. \right) = (bx^n + a)^{m+1} \times \left(\frac{1}{(mn+b+1)b} \times x^{b+1-n} - \right. \\
 &\frac{b+1-na}{mn+b+1 \cdot mn+b+1-nb^2} \times x^{b+1-2n} + \\
 &\left. \frac{b+1-n \times b+1-2n \times a^2}{mn+b+1 \times mn+b+1-n \times mn+b+1-2n \times b^3} \times x^{b+1-2n} - \&c. \right) \text{ (L).}
 \end{aligned}$$

The first

first series, which ascends according to the dimensions of x^n , terminates, when $\frac{mn+b+1}{n}$ is a whole negative number; the second, which descends according to the dimensions of the same quantity x^n , terminates when $\frac{b+1}{n}$ is a whole affirmative number. The fluent will terminate both ways when $\frac{b+1}{n}$ is a whole affirmative and m a negative number greater than $\frac{b+1}{n}$.

5. When m is a whole number or $=0$, the fluent $\int \overline{a+bx^n}^m x^b \dot{x}$ can always be found in finite terms of x , or in the above-mentioned finite terms, together with the log. of x ; which appears from reducing $\overline{a+bx^n}^m$ into simple terms $a^m + ma^{m-1}bx^n + m \cdot \frac{m-1}{2} a^{m-2} b^2 x^{2n} + \&c.$, and multiplying them into $x^b \dot{x}$, and finding the fluents of the resulting fluxions: but the series found by the preceding method will not always terminate when m is a whole number, and the series findable as above mentioned; when properly corrected, it may be rendered findable, or, which means the same thing, the series may be made to terminate.

6. When the series which expresses the fluent of a fluxion terminates, we may begin either from one end of the series, or the other; for example, in finding the fluent of the fluxion $\overline{a+bx^n}^m \times x^b \dot{x}$, either assume the series $\overline{a+bx^n}^{m+1} \times (\alpha x^{b+1} + \beta x^{b+2n+1} + \gamma x^{b+2n+1} + \&c.)$ or the series $\overline{a+bx^n}^{m+1} \times (A + Bx^n + Cx^{2n} + \&c.)$; the former, as is before mentioned, terminates
when

when $\frac{mn+b+1}{n}$ is a whole negative number; the latter when $\frac{b+1}{n}$ is a whole affirmative number.

In like manner assume for a descending series $(bx^n + a)^{m+1} \times (\alpha'x^{b+1-n} + \beta'x^{b+1-2n} + \gamma'x^{b+1-3n} + \&c.)$, or $(bx^n + a)^{m+1} \times (Ax^{-m+1n} + Bx^{-m+2n} + Cx^{-m+3n} + \&c.)$; the former will terminate when $\frac{b+1}{n}$ is a whole affirmative; the latter when $\frac{mn+b+1}{n}$ is a whole negative number.

It appears, therefore, that a series will terminate equally by an ascending or descending series; and the end of the one ascending series is the beginning of its correspondent descending one. All these series, which do not terminate, proceed on *in infinitum*; one term in the former series becomes infinite, when $b+1+z n = 0$; and in the latter (L) when $mn+b+1-z n = 0$, z being a whole affirmative number.

10. It has been observed in the *Medit. Analyt.* that if some quantities contained in the given irrational ones are much less or greater than the rest, it may be preferable in the former case to reduce them into serieses not proceeding according to the dimensions of x , but according to the dimensions of those quantities; and in the latter case according to the reciprocal dimensions of them; and particularly so if the fluent or integral of the terms of the resulting serieses can be found in finite terms, or by tables already calculated.

From similar principles to those before given may be found when the resulting series will converge, and when not.

This method will in many problems be useful, when the value of a near approximate is known.

Of this case I shall subjoin a few examples, of which some have been already published in the *Medit. Analyt.*

Ex. 1. Let the sine and cosine of a given arc A be respectively s and c ; then will the sine and cosine of an arc $A + e$, where e bears a very small ratio to A , be respectively $s + \frac{ce}{r} - \frac{s}{1 \cdot 2r^2} e^2 - \frac{ce^3}{1 \cdot 2 \cdot 3r^3} + \frac{s}{1 \cdot 2 \cdot 3 \cdot 4r^4} e^4 + \&c.$ (the signs proceed by pairs alternately negative and affirmative) $= s(1 - \frac{1}{1 \cdot 2r^2} \times e^2 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4r^4} \times e^4 - \&c.) + c(\frac{e}{r} - \frac{e^3}{1 \cdot 2 \cdot 3r^3} + \frac{e^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5r^5} - \&c.)$; and $c - \frac{s}{r} e - \frac{1}{1 \cdot 2r^2} ce^2 + \frac{1}{1 \cdot 2 \cdot 3r^3} \times se^3 + \&c.$ (the signs proceed as before by pairs alternately negative and affirmative) $= c(1 - \frac{1}{1 \cdot 2r^2} e^2 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4r^2} e^4 - \&c.) - \frac{s}{r} (e - \frac{1}{2 \cdot 3r^2} e^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5r^4} e^5 - \&c.)$

This series can also easily be derived from NEWTON'S series by plane trigonometry, and will converge much swifter than the series $A - \frac{A^3}{1 \cdot 2 \cdot 3r^2} + \frac{A^5}{2 \cdot 3 \cdot 4 \cdot 5r^4} - \&c. \&c.$ if e bears a small proportion to A .

Ex. 2. Let e be a small quantity in proportion to t , and the given fluxion $\frac{\dot{t}}{r^2 + (t+e)^2} = \frac{\dot{t}}{r^2 + t^2} - \frac{2te + e^2}{(r^2 + t^2)^2} \dot{t} + \frac{(2te + e^2)^2}{(r^2 + t^2)^3} \times \ddot{t} - \frac{(2te + e^2)^3}{(r^2 + t^2)^4} \times \dot{t} + \&c. = \frac{\dot{t}}{r^2 + t^2} - \frac{2t\dot{t}}{(r^2 + t^2)^2} \times e - \frac{(r^2 + t^2 - 2^2 \times t^2)\dot{t}}{(r^2 + t^2)^3} \times e^2 + 2\dot{t} \times \frac{(2r^2 + t^2 - 2^2 t^2)\dot{t}}{(r^2 + t^2)^4} \times e^3 + \&c. (P)$; the co-efficient of a term $\dot{t} \times e^{2m}$ of this series will be a fraction whose numerator is $\frac{r^2 + t^2}{r^2 + t^2} - m + 1 \cdot \frac{m}{2} \times 2^2 \times t^2 \times r^2 + t^2^{m-1} + m + 2 \cdot \frac{m+1}{2} \cdot \frac{m}{3} \cdot \frac{m-1}{4} \times 2^4 \times t^4 \times r^2 + t^2^{m-2} - m + 3 \cdot \frac{m+2}{2} \cdot \frac{m+1}{3} \cdot \frac{m}{4} \cdot \frac{m-1}{5} \cdot \frac{m-2}{6}$

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$\times 2^6 \times t^0 \times \overline{r^2 + t^2}^{n-3} + \&c.$, and denominator $(r^2 + t^2)^{2m+1}$; and the co-efficient of a term $t \times e^{2m+1}$ will be a fraction whose numerator is $2t \times (\overline{m+1} \cdot \overline{r^2 + t^2}^m - \overline{m+2} \cdot \frac{m+1}{2} \cdot \frac{m}{3} \times 2^2 \times t^2 \times \overline{r^2 + t^2}^{m-1} + \overline{m+3} \cdot \frac{m+2}{2} \cdot \frac{m+1}{3} \cdot \frac{m}{4} \cdot \frac{m-1}{5} \times 2^4 t^4 \times \overline{r^2 + t^2}^{m-2} - \&c.)$ and denominator $(r^2 + t^2)^{2m+2}$; m being a whole number. The continuation of these serieses is too evident to need enunciation, as must generally be the case when the number of factors contained in the successive terms continues the same or increases in arithmetical progression; and the factors themselves increase or diminish in an arithmetical progression: the terms proceed alternately + and - by pairs.

2. 2. Let the given fluxion be $\frac{e}{r^2 + (t+c)^2} = \frac{e}{r^2 + t^2} - \frac{2te + e^2 \times e}{(r^2 + t^2)^2} + \frac{(2te + e^2)^2 \times e}{(r^2 + t^2)^3} - \&c. = \frac{e}{r^2 + t^2} - \frac{2te}{(r^2 + t^2)^2} \times e - \frac{(r^2 + t^2 - 2^2 t^2)e}{(r^2 + t^2)^3} \times e^2 + \&c.$ (Q); this series becomes the same as the preceding by substituting in it e for t ; in this series e is considered as variable, in the preceding t .

The fluent of the former series (P) is $\int \frac{t}{r^2 + t^2} + \frac{e}{r^2 + t^2} - \frac{t}{(r^2 + t^2)^2} \times e^2 - \frac{(r^2 + t^2 - 2^2 t^2)}{3(r^2 + t^2)^3} \times e^3 - \&c.$; the fluent of the latter series (Q) is the same as of the former (P) except the first term $\int \frac{t}{r^2 + t^2}$: the co-efficient of the term e^b in the fluents will be the same as the co-efficient of the term e^{b-1} in the serieses (P) and (Q) divided by b .

This series, when the tangent of a given arc is known, finds the arc whose tangent differs very little from the given tangent.

11. Let t be increased or diminished by a quantity e in (P), where e bears a very small ratio to any root or value of t in

the equations $P=0$ or $\frac{1}{P} = 0$; and the resulting quantity expanded into a series $a + be + ce^2 + de^3 + \&c.$ proceeding according to the dimensions of e , and this series be multiplied respectively into i and e ; and the fluents of the resulting fluxions found; then will the former differ from the latter by $\int at$; for the former will be $\int at + ae + \frac{\dot{a}}{2i} \times e^2 + \frac{\ddot{a}}{2 \cdot 3i^2} \times e^3 + \frac{\ddot{\dot{a}}e^4}{2 \cdot 3 \cdot 4i^3} + \&c.$, and the latter $ae + \frac{\dot{a}}{2i} e^2 + \frac{\ddot{a}e^3}{2 \cdot 3i^2} + \&c.$

2. In the same manner, if more than one variable quantities $x, y, z, \&c.$ are contained in (P), which are increased by small increments or decrements $\alpha, \beta, \gamma, \&c.$, may the increments or decrements of the quantity (P) be deduced from the incremental theorem.

Ex. Let the quantity (P) be $(a + bx^n)^m$, and a, b, x, n , and m be increased by very small increments $\alpha, \beta, \gamma, \delta$, and ϵ ; then will $(a + \alpha + \overline{b + \beta} \times \overline{x + \gamma}^{n+\delta})^{m+\epsilon} = (a + bx^n)^m + (a + bx^n)^m \times \log. \overline{a + bx^n} \times \epsilon + m \times (a + bx^n)^{m-1} \times (\alpha + \beta x^n + b (x^n \delta \times \log. x + nx^{n-1} \gamma)) + \&c.$

The terms are to be so placed, according to the dimensions of the increments or decrements, that the greatest may first occur.

12. Let some compound quantities be increased by any small quantities variable or invariable, but not the variable quantities contained in them; then reduce the given quantity into a series proceeding according to the dimensions of the small quantities, and find the fluent, integral, &c. as required.

Ex. Let the given quantity be $\frac{x^m \dot{x}}{(x^n + a^n + \beta)^r} = \frac{x^m \dot{x}}{(x^n + a^n)^r} - r\beta$

$\frac{r\beta x^m}{(x^n+a^n)^{r+1}} + r \cdot \frac{r+1}{2} \times \frac{\beta^2 x^m}{(x^n+a^n)^{r+2}} - r \cdot \frac{r+1}{2} \cdot \frac{r+2}{3} \times \frac{\beta^3 x^m}{(x^n+a^n)^{r+3}} +$
 &c., where the quantity $x^n + a^n$ is increased by a small quantity β ; then will $\int \frac{x^m \dot{x}}{(x^n+a^n)^r} = \int \frac{x^m \dot{x}}{(x^n+a^n)^r} - \frac{\beta}{n \times a^n} \left(\frac{x^{m+1}}{(x^n+a^n)^r} - \overline{m+1} \right.$
 $\left. - \overline{rn} \times P \right) + \frac{r\beta^2}{2 \cdot n \times a^n} \left(\frac{x^{m+1}}{(x^n+a^n)^{r+1}} - (m+1 - \overline{r+1n}) \times Q \right)$
 $\frac{r \cdot \overline{r+1}\beta^3}{2 \cdot 3 \cdot n \times a^n} \left(\frac{x^{m+1}}{(x^n+a^n)^{r+2}} - (m+1 - \overline{r+2n}) \times R \right) + \frac{r \cdot \overline{r+1} \cdot \overline{r+2} \cdot \beta^4}{2 \cdot 3 \cdot 4 \cdot n \times a^n}$
 $\left(\frac{x^{m+1}}{(x^n+a^n)^{r+3}} - (m+1 - \overline{r+3n}) \times S \right)$ &c. The continuation of the series is evident; the letters P, Q, R, S, &c. denote the fluents $\int \frac{x^m \dot{x}}{(x^n+a^n)^r}$, $\int \frac{x^m \dot{x}}{(x^n+a^n)^{r+1}}$, $\int \frac{x^m \dot{x}}{(x^n+a^n)^{r+2}}$, &c.

From the length of the arc of an hyperbola or ellipse given may be deduced by this series the length of a correspondent arc of an hyperbola or ellipse, of which the equations expressing the relation between its abscissæ and ordinates differ only by very small quantities from the equations expressing the relation between the abscissæ and ordinates of the former.

13. The same principles may be applied to the resolution of algebraical, fluxional, incremental, &c. equations.

Ex. Let LmM , &c. (Tab. IV. fig. 1.) be a given circle, whose center is C and radius (r), and it be required to find an arc LM , so that the area LSM described round a given point S contained between the lines LS , SM , and the arc of the circle LM be equal to a given area (α).

Find an arc $Lm = A$ nearly equal to the arc required LM , of which to radius 1 substitute s for the sine, and c for the cosine, and write e for $mM = LM - Lm$ and b for SC ; then will

$$LM \times \frac{r}{2} \left(\overline{A+e} \times \frac{r}{2} \right) \doteq SC \times \frac{r}{2} \times \text{fin. } s \text{ arc} : LM \left(\frac{b}{2} \times (A+e -$$

$A+e$

$$\frac{A+e^3}{2 \cdot 3r^2} + \frac{A+e^5}{2 \cdot 3 \cdot 4 \cdot 5r^4} \&c.) = \frac{A}{2} \times r \pm \frac{b}{2} \times rs + \frac{r \pm \frac{b}{2}}{2} \times c \times e \mp \frac{b}{2r} \times \left(\frac{1}{2} \times se^2 + \frac{1}{2 \cdot 3r} \times ce^3 - \frac{1}{2 \cdot 3 \cdot 4r^2} \times se^4 - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5r^3} \times ce^5 + \&c. \right) = \alpha;$$

in this equation for $\frac{2\alpha - Ar \mp b \times rs}{r \pm bc}$ substitute π , and for $r \pm bc$ write t , and the equation resulting will be $e \mp \frac{b}{2tr} se^2 \mp \frac{b}{1 \cdot 2 \cdot 3tr^2} \times ce^3 \pm \frac{b}{1 \cdot 2 \cdot 3 \cdot 4tr^3} \times se^4 \pm \frac{b}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5tr^4} \times ce^5 \mp \&c. = \pi$. From it find e in terms proceeding according to the dimensions of π , and there results $e = \pi \pm \frac{bs}{1 \cdot 2tr} \times \pi^2 \pm \frac{3b^2s^2 \pm bct}{2 \cdot 3t^2r^2} \times \pi^3 \pm \&c.$

This method of resolving KEPLER's and other problems of cutting a given area described round a point, whether focus or not, in a circle, when an approximate sufficiently near to the area to be found is given, will converge as swift as any known method.

The resolution of this problem may be deduced somewhat different by the following methods. Let the letters b, r, α , denote the same quantities as before, and s be the sine of the arc Lm to radius 1, and $s + o$ the sine of the arc LM nearly = Lm ; then will $\frac{r^2}{2} \int \frac{s}{\sqrt{(1-s+o^2)}} \pm \frac{br}{2} \times \sqrt{s+o} = \alpha$; reduce the fluxion $\frac{s}{\sqrt{(1-s+o^2)}}$ into a series $P s + Q o s + R o^2 s + S o^3 s + \&c.$ and find the fluents of the fluxions $Q s, R s, S s, \&c.$, which let be $B, C, D, \&c.$, there will result the equation $o + \frac{C}{t} o^2 + \frac{D}{t} o^3 + \&c. = \frac{2\alpha \mp brs - r \times A}{t} = \pi$, where $t = Br^2 \pm br$ and $A = \text{arc } Lm$; find o in terms of a series proceeding to the dimensions of π , and consequently $s + o$ the sine of the arc LM required.

From

From similar principles may be found the tangent of the arc required from the equation $\frac{r^2}{2} \int \frac{i}{1+(t+i)^2} \pm \frac{t+i \times r \times b}{2\sqrt{(1+t+i)^2}} = \alpha$, where the tangent t to rad. 1 of the arc $Lm = A$ is given. Let the fluent of the fluxion $\left(\frac{i}{1+(t+i)^2}\right) = A + Bi + Ci^2 + Di^3 + \&c.$; and $\frac{t+i \times b}{\sqrt{(1+t+i)^2}} = A' + B'i + C'i^2 + D'i^3 + \&c.$, the equation will become $i + \frac{rC \pm C'}{rB \pm B'} i^2 + \frac{rD \pm D'}{rB \pm B'} \times i^3 + \&c. = \frac{2\alpha - r^2 A \pm rA'}{(r^2 B \pm rB')} = \pi$; from this equation investigate $i = \pi + P\pi^2 + Q\pi^3 + \&c.$, and thence $t+i$ the tangent required.

In like manner may be found the secant, cosine, &c. of the arc required.

14. The same principles may be applied to cut an area described round any given point in a given curve equal to an area α .

Let x be the abscifs and y the ordinate of the given curve, and b the distance of the beginning of the abscifs from the given points, and let (A) be the area of the curve described round the point S , when the abscifs is x , which differs very little from the given area (α) ; to find the value $x+e$ of the abscifs, when the area $= \alpha$.

Let $y=X$ a function of x , and in X for x write $x+e$, and reduce the resulting quantity (X') into a series $X + Be + Ce^2 + De^3 + \&c.$ proceeding according to the dimensions of e ; then will the area $\int yx = \int Xx + e \int Bx + e^2 \int Cx + e^3 \int Dx + \&c. = A + be + ke^2 + le^3 + \&c.$, and consequently $A + be + ke^2 + le^3 + \&c. \pm (b \pm x + e) \times \frac{y}{2} = A + be + ke^2 + le^3 + \&c. + \frac{1}{2}(b \pm x + e) \times (X + Be + Ce^2 + De^3 + \&c.) = A + \frac{1}{2}(b \pm x) \times X + (b + \frac{1}{2}b \pm x) \times$

$B + \frac{1}{2}X)e + (k + \frac{1}{2}(b \pm x)C + \frac{1}{2}B)e^2 + (l + \frac{1}{2}\overline{b \pm x} \times D + \frac{1}{2}C)e^3 + \dots = \alpha$; find the value of e in a series proceeding according to the dimensions of $\frac{\alpha - A - \frac{1}{2}(b \pm x) \times X}{b + \frac{1}{2}b \pm x \times B + \frac{1}{2}X}$, and $x + e$ will be the absciss required; X denotes the value of y , when the absciss $= x$.

From the same principles may the ordinate y , &c. be found.

This problem may be resolved in the same manner, when X denotes an infinite series deduced from an equation expressing the relation between the absciss and ordinate of the given curve.

If the given area α be the difference between two areas SPM (α') and SPQ $= \beta$ (fig. 2.); for α substitute $\alpha' - \beta$, and the operation will be the same as the preceding.

15. 1. Given any equations, of which the increments of the quantities contained in them can be found from each other, and given approximate values of each of the unknown quantities, which nearly correspond to each other; to find approximations, which differ less from the quantities themselves than the given ones.

Suppose each of the given approximate values to be increased or diminished by small increments or decrements, as e, o, i , &c. which are the approximations to be found; and from the given find the equations resulting from this hypothesis; and from these may be deduced, by simple equations, the approximations sought e, i, o , &c. by neglecting in them all the powers of e, o, i , &c. except the simple ones, and all the products of them multiplied into each other; and consequently the equations deduced will contain only given quantities and simple powers of the unknown e, i, o , &c. to be found.

2. When two or more (n) values of one (x) of the unknown quantities are nearly equal to its given approximate; then the

equation which finds the approximate to x will be a quadratic or an equation of n dimensions.

3. The approximations found by this method will converge more or less, according as the approximations given are more or less near to one value of each of the unknown quantities than to the remaining ones, &c.

These principles were printed in the *Medit. Algebr.* in the years 1768 and 1769.

L E M M A.

15. Let kab (fig. 3.) be a circle, whose center is o ; and Po perpendicular to the plane of the circle; and the force of any corpuscle in the circle on the particle P vary as the corpuscle divided by the (n) power of its distance from the particle; to find the attraction of the circle kab on the particle P .

From the supposita the force of any ring contained between the nearest concentric circles lcd and hef of which the center is o , on the particle P , will be as the area of the ring divided by the n th power of the distance Pb ; and consequently if $Po = A$, $ob = v$, $bl = \dot{v}$, and $p =$ periphery of a circle of which the radius is 1; the attraction of the ring on P will be as

$$\frac{p \times v \dot{v}}{(A^2 + v^2)^{\frac{n}{2}}}, \text{ and the force of it in the direction } Po \text{ as } \frac{Po}{Pb} \times$$

$$\frac{p v \dot{v}}{(A^2 + v^2)^{\frac{n}{2}}} = \frac{p A v \dot{v}}{(A^2 + v^2)^{\frac{n+1}{2}}}; \text{ of which fluxion the fluent is}$$

$$\frac{p A}{1 - n \times (A^2 + v^2)^{\frac{n-1}{2}}}; \text{ and thence the force of the circle } kab \text{ on the}$$

$$\text{particle } P \text{ will be as } \frac{p \times A}{1 - n P k^{n-1}} - \frac{p \times A}{1 - n P O^{n-1}} \left(\frac{p}{1 - n A^{n-2}} \right).$$

Cor. 1. The force of the area contained between the two circles *kab* and *lcd* on the particle P will be as $\frac{pA}{1 - nPk^{n-1}} - \frac{pA}{1 - nPl^{n-1}}$.

15. Let a solid be generated by the rotation of a given curve round a line in the axis *Po* or *Po* produced, and *A* the distance of P from *o*, and *x*=distance of any circle generated by the rotation of a point in the curve round the axis *Po* from *o*; and *y* the ordinate to the given curve, of which the absciss is *x*; and the same things be supposed as before; then will the attraction of the solid on the point P vary as

$$\int \frac{p \times \overline{A-x} \times \dot{x}}{n-1(A-x)^{n-1}} \left(\frac{p\dot{x}}{n-1(A-x)^{n-2}} \right) - \int \frac{p \times \overline{A-x} \times \dot{x}}{n-1 \times (\overline{A-x^2} + y^2)^{\frac{n-1}{2}}}$$

Let the given curve be a circle, of which the center is *o*, and radius *t*, and consequently $y^2 = t^2 - x^2$, and $\int \frac{p\dot{x}}{n-1(A-x)^{n-2}}$

$$- \int \frac{p \times \overline{A-x} \times \dot{x}}{n-1 \cdot (\overline{A-x^2} + y^2)^{\frac{n-1}{2}}} = \frac{p}{n-1 \cdot n-3 \times (A-x)^{n-3}} - \int \frac{p\dot{x} \times \overline{A-x}}{n-1 \times (A^2 + t^2 - 2Ax)^{\frac{n-1}{2}}} = \frac{p}{n-1 \cdot n-3(A-x)^{n-3}} - \frac{p}{n-1} \left(\frac{1}{n-3} + \frac{A^2 + t^2}{n-5 \cdot n-3A^2} - \frac{x}{n-5A} = \frac{n-4A^2 + t^2 - n-3Ax}{n-5 \cdot n-3 \times A^2} \right) \times (A^2 + t^2 - 2Ax)^{\frac{3-n}{2}}$$

By substituting $-t$ and t for *x* in the preceding expression the attraction of the whole globe on the particle P will be found $\frac{p}{n-1} \left(\frac{A^2 + n-3At + t^2}{n-5 \cdot n-3 \times A^2} \times \overline{A+t}^{3-n} \text{ (L)} - \frac{A^2 - n-3At + t^2}{n-5 \cdot n-3 \times A^2} \times \overline{A-t}^{3-n} \text{ (H)} \right)$.

16. If the particle be situated within the globe, and consequently *A* less than *t* or $-t$, and *n* be an odd number negative

or affirmative; or fraction, of which the numerator is odd; then the preceding resolution will be physically just; for in this case, if the attraction on one side of the particle be affirmative, the attraction on the other will mathematically be negative, that is, physically opposite: but if n be an even number, or fraction, of which the numerator is an even and denominator an odd number, the mathematical solution will not agree to the physical; for in the former the force on both sides will be affirmative, in the latter the forces will be opposite, and therefore physically the force in this case will vary as $L + H$, and not as $L - H$, which is the force in both the cases when A is greater than t and $-t$. The same may be applied to the more general resolution.

17. Let ABCD (fig. 4.) be a globe, of which the diameters AB and CD are situated at right angles to each other, and AHBL be a spheroid generated by the revolution of an ellipse on its axis AB, to which let HL nearly equal to $CD = AB$ be the conjugate, and P a point in the axis BA produced; to find the attraction of the ring contained between the globe and the spheroid on the point P, on the supposition that the force of any corpuscle in the ring on the particle P varies as the magnitude of the corpuscle directly, and the n th power of its distance from the corpuscle inversely.

Let $AB = CD = 2t$, $CD = HL (2c) + 2CH = 2c + 2e$, and consequently $OH = c = t - e$, where e has a very small ratio to t , $oP = A$, $po = x$, and pM parallel to $CD = y$; then, by the preceding lemma, the attraction of the circle whose radius

is pM on the point P will be $\frac{\rho \times A - x}{n - 1 P \rho^{n-1}} - \frac{\rho \times A - x}{n - 1 PM^{n-1}}$; and in

like manner the attraction of the circle, whose radius is pm ,

will be $\frac{p \times \overline{A-x}}{n-1 \cdot Pp^{n-1}} - \frac{p \times \overline{A-x}}{n-1 \cdot Pm^{n-1}}$, and consequently the attraction of the ring contained between the two circles is $\frac{p \times \overline{A-x}}{n-1 \cdot Pm^{n-1}}$

$$- \frac{p \times \overline{A-x}}{n-1 \cdot Pm^{n-1}} = \frac{p}{n-1} \times \left(\frac{A-x}{\left(\overline{A-x^2} + \frac{t-e^2}{t^2} \times \overline{t^2-x^2} \right)^{\frac{n-1}{2}}} - \frac{A-x}{\left(\overline{A-x^2} + t^2-x^2 \right)^{\frac{n-1}{2}}} \right) = \frac{p}{n-1} \times \frac{\overline{n-1} \times \overline{A-x} \times \left(t - \frac{x^2}{t} \right)}{\left(A^2 + t^2 - 2Ax \right)^{\frac{n+1}{2}}} \times e + \&c. \times e^2 + \&c. = p \times \left(t - \frac{x^2}{t} \right) \times \frac{A-x}{\left(A^2 + t^2 - 2Ax \right)^{\frac{n+1}{2}}} \times e \text{ nearly, which multi-}$$

plied into \dot{x} , and the fluent of the resulting fluxion found, it will be $p \times \left(A^2 + t^2 - 2Ax \right)^{\frac{1-n}{2}} \times \left(\left(\frac{1}{n-7 \times tA} = d \right) x^3 - \left(\frac{1}{n-5t} + \frac{3d \times \overline{A^2+t^2}}{n-5A} = c \right) x^2 - \left(\frac{t}{n-3A} + 2c \times \frac{A^2+t^2}{n-3 \times A} = b \right) x + \frac{t}{n-1} - \frac{b \times \overline{A^2+t^2}}{n-1A} \right) e(M).$

If the attraction of the whole ring contained between the sphere and spheroid be required, substitute in the fluent (M) for x , t and $-t$, and proceed as in the preceding cases; in like manner may be found the attraction of the ring contained between any two values of x .

18. The same principles may be applied to find the attraction of the above-mentioned ring, when the line Pp is not perpendicular to the circles pM , pm , CD , &c., and does not cut the diameters CD , $M'M$, &c. into two equal parts. They may be further applied for finding the attraction of rings contained between any other given solids, of which the equations differ by very small quantities from each other; for example,

between two spheroids, of which the axes in the one do not differ much from correspondent axes in the other, in which case the fluents found of the following fluxions may be useful.

$$\begin{aligned}
 \text{I. } \int \frac{\dot{x}}{(t^2-x^2)^{\frac{n-1}{2}}} &= \frac{1}{n-3t^2} \times \frac{x}{(t^2-x^2)^{\frac{n-3}{2}}} + \frac{n-4}{n-3 \cdot n-5t^4} \times \frac{x}{(t^2-x^2)^{\frac{n-5}{2}}} \\
 + \frac{n-4 \cdot n-6}{n-3 \cdot n-5 \cdot n-7t^6} &\times \frac{x}{(t^2-x^2)^{\frac{n-7}{2}}} + \frac{n-4 \cdot n-6 \cdot n-8}{n-3 \cdot n-5 \cdot n-7 \cdot n-9t^8} \times \\
 \frac{x}{(t^2-x^2)^{\frac{n-9}{2}}} &\dots \dots \frac{n-4 \cdot n-6 \cdot n-8 \dots 3}{n-3 \cdot n-5 \cdot n-7 \cdot n-9 \dots 2t^{n-3}} \times \frac{x}{t-x^2} + \\
 \frac{n-4 \cdot n-6 \cdot n-8 \dots 3 \cdot 1}{n-3 \cdot n-5 \cdot n-7 \cdot n-9 \dots 2t^{n-3}} &\times \int \frac{\dot{x}}{t^2-x^2} \text{ if } n \text{ be an odd num-} \\
 \text{ber; but} &
 \end{aligned}$$

$$\begin{aligned}
 \text{2. } \int \frac{\dot{x}}{(t^2-x^2)^{\frac{n-1}{2}}} &= \frac{1}{n-3t^2} \times \frac{x}{(t^2-x^2)^{\frac{n-3}{2}}} + \frac{n-4}{n-3 \cdot n-5t^4} \times \frac{x}{(t^2-x^2)^{\frac{n-5}{2}}} \\
 + \frac{n-4 \cdot n-6}{n-3 \cdot n-5 \cdot n-7t^6} &\times \frac{x}{(t^2-x^2)^{\frac{n-7}{2}}} \dots \dots + \\
 \frac{n-4 \cdot n-6 \cdot n-8 \cdot n-10 \dots 4 \cdot 2}{n-3 \cdot n-5 \cdot n-7 \cdot n-9 \cdot n-11 \dots 3 \cdot 1t^{n-2}} &\times \frac{x}{(t^2-x^2)^{\frac{1}{2}}}, \text{ if } n \text{ be} \\
 \text{even.} &
 \end{aligned}$$

These principles may be applied to the finding approximations in very many philosophical problems.

Cor. 1. From hence may be deduced the subsequent arithmetical theorems.

$$\begin{aligned}
 \text{I. } \frac{1}{2m-1} + \frac{2m-2}{2m-1 \cdot 2m-3} + \frac{2m-2 \cdot 2m-4}{2m-1 \cdot 2m-3 \cdot 2m-5} + \\
 \frac{2m-2 \cdot 2m-4 \cdot 2m-6}{2m-1 \cdot 2m-3 \cdot 2m-5 \cdot 2m-7} \dots \dots \frac{2m-2 \cdot 2m-4 \cdot 2m-6 \dots 2}{2m-1 \cdot 2m-3 \cdot 2m-5 \cdot 2m-7 \dots 3 \cdot 1} \\
 = 1; \text{ or, which is the same, } \frac{1}{2m-3} + \frac{2m-4}{2m-3 \cdot 2m-5} + \\
 2m-4
 \end{aligned}$$

$$\frac{2m-4 \times 2m-6}{2m-3 \times 2m-5 \times 2m-7} \dots + \frac{2m-4 \cdot 2m-6 \cdot 2m-8 \dots 4 \times 2}{2m-3 \cdot 2m-5 \cdot 2m-7 \cdot 2m-9 \dots 3 \times 1}$$

$$= 1, \text{ or } \frac{1}{2m-5} + \frac{2m-6}{2m-5 \cdot 2m-7} \dots + \frac{2m-6 \cdot 2m-8 \dots 4 \times 2}{2m-5 \cdot 2m-7 \cdot 2m-9 \dots 3 \times 1}$$

$$= 1, \text{ \&c.}$$

$$2. \frac{1}{2m-2} + \frac{2m-3}{2m-2 \cdot 2m-4} + \frac{2m-3 \cdot 2m-5}{2m-2 \cdot 2m-4 \cdot 2m-6} +$$

$$\frac{2m-3 \cdot 2m-5 \cdot 2m-7}{2m-2 \cdot 2m-4 \cdot 2m-6 \cdot 2m-8} \dots +$$

$$\frac{2m-3 \cdot 2m-5 \cdot 2m-7 \cdot 2m-9 \dots 7 \cdot 5}{2m-2 \cdot 2m-4 \cdot 2m-6 \cdot 2m-8 \cdot 2m-10 \dots 6 \cdot 4} + 2 \times$$

$$\frac{2m-3 \cdot 2m-5 \cdot 2m-7 \cdot 2m-9 \dots 3}{2m-2 \cdot 2m-4 \cdot 2m-6 \cdot 2m-8 \cdot 2m-10 \dots 2} = 1, \text{ or}$$

$$\frac{1}{2m-4} + \frac{2m-5}{2m-4 \cdot 2m-6} + \frac{2m-5 \cdot 2m-7}{2m-4 \cdot 2m-6 \cdot 2m-8} \dots +$$

$$\frac{2m-5 \cdot 2m-7 \cdot 2m-9 \dots 5}{2m-4 \cdot 2m-6 \cdot 2m-8 \dots 4} + 2 \times$$

$$\frac{2m-5 \cdot 2m-7 \cdot 2m-9 \cdot 2m-11 \dots 5 \cdot 3}{2m-4 \cdot 2m-6 \cdot 2m-8 \cdot 2m-10 \cdot 2m-12 \dots 4 \cdot 2} = 1, \text{ \&c.}$$

3. By expanding the terms of the preceding fluents may be deduced the arithmetical theorems, viz.

$$1. \frac{1}{n-3} \times \frac{n-4}{n-3} \times \frac{n-5}{n-3} \times \frac{n-6}{n-3} \times \frac{n-7}{n-3} \times \frac{n-8}{n-3} \times \frac{n-9}{n-3} \dots \times \frac{n+m}{n-3} + \frac{n-4}{n-3 \cdot n-5}$$

$$\times \frac{n-6}{n-3 \cdot n-5 \cdot n-7} + \frac{n-6 \cdot n-8}{n-3 \cdot n-5 \cdot n-7 \cdot n-9}$$

$$\times \frac{n-8}{n-3 \cdot n-5 \cdot n-7 \cdot n-9} + \frac{n-8 \cdot n-10}{n-3 \cdot n-5 \cdot n-7 \cdot n-9 \cdot n-11} \dots$$

$$+ \dots + \frac{n-4 \cdot n-6 \cdot n-8 \cdot n-10 \dots 3}{n-3 \cdot n-5 \cdot n-7 \cdot n-9 \cdot n-11 \dots 2} \times 2 \cdot 4 \cdot 6 \cdot 8 \dots$$

$$m+5 + \frac{n-4 \cdot n-6 \cdot n-8 \cdot n-10 \dots 3}{n-3 \cdot n-5 \cdot n-7 \cdot n-9 \cdot n-11 \dots 2} \times 2 \times 4 \times 6 \times 8 \dots$$

$m+5$

$$\overline{m+5} \times \frac{1}{m+6} = \overline{n-1} \cdot \overline{n+1} \cdot \overline{n+3} \cdot \overline{n+5} \dots \overline{n+m+2} \times \frac{1}{m+6}$$

if n be an odd number.

$$\begin{aligned} & 2. \frac{1}{n-3} \times \overline{n-3} \cdot \overline{n-1} \cdot \overline{n+1} \cdot \overline{n+3} \dots \overline{n+2m+1} + \\ & \frac{\overline{n-4}}{n-3 \cdot n-5} \times \overline{n-5} \cdot \overline{n-3} \cdot \overline{n-1} \cdot \overline{n+1} \dots \overline{n+2m-1} + \\ & \frac{\overline{n-4} \cdot \overline{n-6}}{n-3 \cdot n-5 \cdot n-7} \times \overline{n-7} \cdot \overline{n-5} \cdot \overline{n-3} \cdot \overline{n-1} \dots \overline{n+2m-3} + \\ & \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8}}{n-3 \cdot n-5 \cdot n-7 \cdot n-9} \times \overline{n-9} \cdot \overline{n-7} \cdot \overline{n-5} \cdot \overline{n-3} \dots \overline{n+2m-5} \\ & \dots + \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \dots 2}{n-3 \cdot n-5 \cdot n-7 \cdot n-9 \dots 3 \times 1} \times 1 \cdot 3 \cdot 5 \cdot 7 \dots \overline{2m+5} = \\ & \overline{n-1} \cdot \overline{n+1} \cdot \overline{n+3} \cdot \overline{n+5} \dots \overline{n+2m+3} \times \frac{1}{2m+7}, \text{ if } n \text{ be an} \end{aligned}$$

even number.

4. Arithmetical theorems, fomewhat different, may be deduced from taking the fluxions of the preceding fluents, and reducing the fractions resulting to a common denominator by multiplying them into $t^2 - x^2$, $(t^2 - x^2)^2$, $(t^2 - x^2)^3$, &c.

$$\begin{aligned} & 1. \frac{1}{n-3} \times \overline{n-4} + \frac{\overline{n-4}}{n-3 \cdot n-5} \times \overline{n-7} + \frac{\overline{n-4} \cdot \overline{n-6}}{n-3 \cdot n-5 \cdot n-7} \times \overline{n-10} \\ & + \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8}}{n-3 \cdot n-5 \cdot n-7 \cdot n-9} \times \overline{n-13} \dots + \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \dots 3}{n-3 \cdot n-5 \cdot n-7 \cdot n-9 \dots 2} \\ & \times \frac{7-n}{2} - \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \cdot \overline{n-10} \dots 3}{n-3 \cdot n-5 \cdot n-7 \cdot n-9 \cdot n-11 \dots 2} \times \frac{n-3}{2} = 0, \text{ if } n \text{ be} \end{aligned}$$

odd.

$$\begin{aligned} & 2. \frac{1}{n-3} \times \overline{n-4} + \frac{\overline{n-4}}{n-3 \cdot n-5} \times \overline{n-7} + \frac{\overline{n-4} \cdot \overline{n-6}}{n-3 \cdot n-5 \cdot n-7} \times \overline{n-10} \\ & + \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8}}{n-3 \cdot n-5 \cdot n-7 \cdot n-9} \times \overline{n-13} \dots + \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \dots 2}{n-3 \cdot n-5 \cdot n-7 \cdot n-9 \dots 3 \times 1} \\ & \times \frac{4-n}{2} = 0, \text{ if } n \text{ be even; or more general, if } n \text{ be even,} \end{aligned}$$

$$\begin{aligned}
 3. & \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \dots \overline{n-2m-2}}{\overline{n-3} \cdot \overline{n-5} \cdot \overline{n-7} \cdot \overline{n-9} \dots \overline{n-2m-3}} \times \overline{n-2m-4} + \\
 & \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \cdot \overline{n-10} \dots \overline{n-2m-4}}{\overline{n-3} \cdot \overline{n-5} \cdot \overline{n-7} \cdot \overline{n-9} \cdot \overline{n-11} \dots \overline{n-2m-5}} \times (\overline{m+1} \times \overline{n-2m-6-1}) \\
 & + \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \cdot \overline{n-10} \cdot \overline{n-12} \dots \overline{n-2m-6}}{\overline{n-3} \cdot \overline{n-5} \cdot \overline{n-7} \cdot \overline{n-9} \cdot \overline{n-11} \cdot \overline{n-13} \dots \overline{n-2m-7}} \times \\
 & \quad \left(\overline{m+1} \cdot \frac{\overline{m+2}}{2} \times \overline{n-2m-8} - \overline{m+2} \right) + \\
 & \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \cdot \overline{n-10} \cdot \overline{n-12} \cdot \overline{n-14} \dots \overline{n-2m-8}}{\overline{n-3} \cdot \overline{n-5} \cdot \overline{n-7} \cdot \overline{n-9} \cdot \overline{n-11} \cdot \overline{n-13} \cdot \overline{n-15} \dots \overline{n-2m-9}} \\
 & \times \left(\overline{m+1} \cdot \frac{\overline{m+2}}{2} \cdot \frac{\overline{m+3}}{3} \times \overline{n-2m-10} - \overline{m+2} \cdot \frac{\overline{m+3}}{2} \right) + \\
 & \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \cdot \overline{n-10} \dots \overline{n-2m-10}}{\overline{n-3} \cdot \overline{n-5} \cdot \overline{n-7} \cdot \overline{n-9} \cdot \overline{n-11} \dots \overline{n-2m-11}} \times \left(\overline{m+1} \cdot \frac{\overline{m+2}}{2} \cdot \frac{\overline{m+3}}{3} \cdot \right. \\
 & \quad \left. \frac{\overline{m+4}}{4} \times \overline{n-2m-12} - \overline{m+2} \cdot \frac{\overline{m+3}}{2} \cdot \frac{\overline{m+4}}{3} \right) \dots \dots + \\
 & \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \dots \overline{2}}{\overline{n-3} \cdot \overline{n-5} \cdot \overline{n-7} \dots 3 \times 1} \times \overline{-m+2} \cdot \frac{\overline{m+3}}{2} \cdot \frac{\overline{m+4}}{3} \dots \dots \frac{\overline{n-4}}{\overline{n-2m-6}} \\
 & = 0.
 \end{aligned}$$

$$\begin{aligned}
 4. & \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \dots \overline{n-2m-2}}{\overline{n-3} \cdot \overline{n-5} \cdot \overline{n-7} \cdot \overline{n-9} \dots \overline{n-2m-3}} \times \overline{n-2m-4} + \\
 & \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \dots \overline{n-2m-4}}{\overline{n-3} \cdot \overline{n-5} \cdot \overline{n-7} \cdot \overline{n-9} \dots \overline{n-2m-5}} \times (\overline{m+1} \times \overline{n-2m-6-1}) + \\
 & \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \dots \overline{n-2m-6}}{\overline{n-3} \cdot \overline{n-5} \cdot \overline{n-7} \cdot \overline{n-9} \dots \overline{n-2m-7}} \times \left(\overline{m+1} \cdot \frac{\overline{m+2}}{2} \times \overline{n-2m-8} - \right. \\
 & \quad \left. \overline{m+2} \right) + \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \dots \overline{n-2m-8}}{\overline{n-3} \cdot \overline{n-5} \cdot \overline{n-7} \cdot \overline{n-9} \dots \overline{n-2m-9}} \times \left(\overline{m+1} \cdot \frac{\overline{m+2}}{2} \times \right. \\
 & \quad \left. \frac{\overline{m+3}}{3} \times \overline{n-2m-10} - \overline{m+2} \cdot \frac{\overline{m+3}}{2} \right) + \dots \dots + \\
 & \frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \dots \overline{5} \times 3}{\overline{n-3} \cdot \overline{n-5} \cdot \overline{n-7} \cdot \overline{n-9} \dots 4 \times 2} \times \left(\overline{m+1} \cdot \frac{\overline{m+2}}{2} \cdot \frac{\overline{m+3}}{3} \dots \frac{\overline{n-5}}{\overline{n-2m-5}} - \right. \\
 & \quad \left. \overline{m+2} \cdot \frac{\overline{m+3}}{2} \cdot \frac{\overline{m+4}}{3} \dots \frac{\overline{n-5}}{\overline{n-2m-7}} \right) -
 \end{aligned}$$

$$\frac{\overline{n-4} \cdot \overline{n-6} \cdot \overline{n-8} \dots \overline{5 \times 3}}{n-3 \cdot n-5 \cdot n-7 \cdot n-9 \dots 4 \times 2} \times \overline{m+2} \cdot \frac{m+3}{2} \times \frac{m+4}{3} \dots$$

$$\frac{n-3}{n-2m-5} = 0, \text{ if } n \text{ be an odd number.}$$

Many more arithmetical theorems may be deduced from the fluents of these and other fluxions by similar methods, which cannot, without some difficulty, be found from the common methods of finding the fums of series.

This method of finding approximations to the areas and lengths of curves, fluents of fluxions, and fums of series, &c. of which the equations, to their increments, fluxions, &c. are given from the areas of curves, fluents, &c. of which the equations to their increments, &c. differ by very small quantities from the given equations was published in the Meditat. Analyt. near twenty years ago.

I shall conclude this Paper with two theorems of some little use in the doctrine of chances.

T H E O R E M I.

$$H = \overline{a+b} \times \overline{a+b-1} \cdot \overline{a+b-2} \cdot \overline{a+b-3} \dots \overline{a+b-n+1}$$

$$= a \cdot \overline{a-1} \cdot \overline{a-2} \dots \overline{a-n+1} + n \cdot a \cdot \overline{a-1} \cdot \overline{a-2} \dots a \cdot \overline{n+2}$$

$$\times b + n \cdot \frac{n-1}{2} \times a \cdot \overline{a-1} \cdot \overline{a-2} \dots \overline{a-n+3} \times b \cdot \overline{b-1} + n \cdot$$

$$\frac{n-1}{2} \cdot \frac{n-2}{3} a \cdot \overline{a-1} \cdot \overline{a-2} \dots \overline{a-n+4} \times b \times b \cdot \overline{b-1} \cdot \overline{b-2} +$$

$$\dots + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \dots \frac{n-l+1}{l} (L) a \cdot \overline{a-1} \cdot \overline{a-2}$$

$$\cdot \overline{a-3} \dots \overline{a-n+l+1} \times b \cdot \overline{b-1} \cdot \overline{b-2} \dots \overline{b-l+1} + \dots +$$

$$n \cdot \frac{n-1}{2} \overline{a \cdot a-1 \cdot b \cdot b-1 \cdot b-2 \dots b-n+3} + na \cdot \overline{b \cdot b-1 \cdot b-2 \dots b-n+2} + \overline{b \cdot b-1 \cdot b-2 \dots b-n+1}$$

If for $a + b - 1, a + b - 2, a + b - 3, \&c., a - 1, a - 2, \&c., b - 1, b - 2, \&c.$ be substituted respectively $a + b - x, a + b - 2x, a + b - 3x, \&c., a - x, a - 2x, a - 3x, \&c., b - x, b - 2x, b - 3x, \&c.$, the resulting equation will equally be just; and, lastly, if for x be substituted 0, it will become the binomial theorem.

Cor. If there are two different events A and B, of which the numbers are respectively a and b , and their chances of happening also as a and b ; and if A's happen, let the whole number $(a + b)$ and also the number of A's be diminished by x , and in the same manner of B's happening, and so on; then will the chance of A's happening $n - l$ times, and B's happening l times in n trials be $L \times a \cdot \overline{a - x \cdot a - 2x \dots a - (n - l - 1)x} \times b \cdot \overline{b - x \cdot b - 2x \dots b - (l - 1)x}$ divided by H.

In a similar manner may be found, 1. the chance of A's happening between b and k times; and, 2. the chance of A's happening (b) to B's happening (k) times; 3. of A's and B's happening respectively b and k times more than the other; 4. the chance of A's happening an even to its happening an odd number of times, &c. in (n) trials, &c. &c. &c.

T H E O R E M II.

$$H = \overline{a + b + c + d + \&c.} \times \overline{a + b + c + d + \&c. - x} \times \overline{a + b + c + d + \&c. - 2x} \dots \overline{a + b + c + d + \&c. - (n - 1)x} = a \cdot \overline{a - x \cdot a - 2x \dots a - n - 1x} + n \cdot a \cdot \overline{a - x \cdot a - 2x \dots a - n - 2x} \times \overline{b + c + d + \&c.} +$$

$$+ n \cdot \frac{n-1}{2} \cdot a \cdot a-x \cdot a-2x \dots a-n-3x \times (b \cdot b-x + c \times c-x$$

$$+ d \cdot d-x + \&c. + 2bc + 2bd + 2cd + \&c.) + \dots + L \times (a \cdot$$

$$a-x \cdot a-2x \dots a-l-1x \times b \cdot b-x \cdot b-2x \dots b-m-1x$$

$$\times c \cdot c-x \cdot c-2x \dots c-p-1x \times d \cdot d-x \cdot d-2x \dots$$

$$d-q-1x \times \&c. = K) + \&c., \text{ where } L = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \dots$$

$$\frac{n-l+1}{l} \times n-l \cdot \frac{n-l-1}{2} \cdot \frac{n-l-2}{3} \dots \frac{n-l-m+1}{m} \times n-l-m$$

$$\frac{n-l-m-1}{2} \cdot \frac{n-l-m-2}{3} \dots \frac{n-l-m-p+1}{p} \times n-l-m-p$$

$$\frac{n-l-m-p-1}{2} \cdot \frac{n-l-m-p-2}{3} \dots \frac{n-l-m-p-q+1}{q} \times \&c. \text{ which is}$$

the same as the co-efficient of the term $a^l \times b^m \times c^p \times d^q \times \&c.$ in the multinomial $a + b + c + d + \&c.$ raised to the power n .

The chance of any number of events A, B, C, D, &c. of which the numbers are $a, b, c, d, \&c.$ happening $l, m, p, q, \&c.$ times respectively in a similar manner to A's and B's happening in the preceding case will be $\frac{L \times K}{H}$.

All the propositions mentioned as immediately deducible from the preceding theorem may, *mutatis mutandis*, with the same ease be applied to more events A, B, C, D, &c.

If for $a, b, c, d, \&c.$ be substituted the same letters, increased or diminished by any given quantities, the resulting equation will be equally true.

Fig 1.

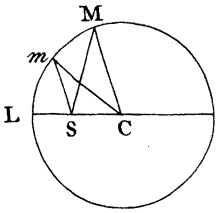


Fig 2.

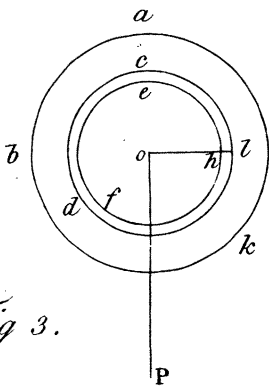
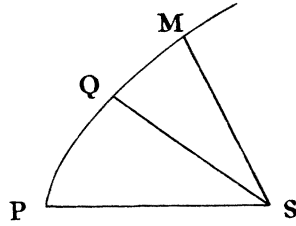


Fig 3.

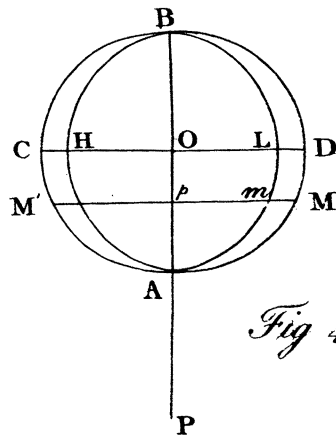


Fig 4.